Lecture 9
Random Variables 2

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Bivariate Joint Distribution

- Let be two discrete random variables on same sample space.
- Probability distribution .
- The distribution function which is denoted by

- .
- Properties of DF: 1.

2. should be a non decreasing function.

3.

Probability Distribution:

Joint pd:

Then will be considered as pd.

Marginal pd for :

Marginal pd for :

Conditional Probability Functions:

- For two discrete random variables and with joint pd then

  and

.

- Two random variables and will be independent if

Example 7.6 An urn contains 3 black, 2 red and 3 green balls and 2 balls are selected at ra
from it. If X is the number of black balls and Y is the number of red balls selected, then find

i) the joint probability function $f(x, y)$;

ii) $P(X + Y \leq 1)$;

iii) the marginal p.d. $g(x)$ and $h(y)$;

iv) the conditional p.d. $f(x \mid 1)$,

v) $P(X=0 \mid Y = 1)$, and

vi) Are X and Y independent?

i) The sample space S for this experiment contains $\binom{8}{2} = 28$ sample points. The possibl

of X are 0, 1 and 2, and those for Y are 0, 1 and 2. The values that (X, Y) can take on ar
(0, 1), (1, 0), (1, 1), (0, 2) and (2, 0). We desire to find $f(x, y)$ for each value (x, y).

Now $f(0, 0) = P(X=0 \text{ and } Y=0)$, where the event ($X=0$ and $Y=0$) represents that neith
nor red ball is selected, implying that the 2 selected are green balls. This event th

contains $\binom{3}{0}\binom{2}{0}\binom{3}{2} = 3$ sample points, and

$$f(0,0) = P(X=0 \text{ and } Y=0) = \frac{3}{28}.$$

Again $f(0,1) = P(X=0 \text{ and } Y=1)$

$= P(\text{none is black, 1 is red and 1 is green})$

$$= \frac{\binom{3}{0}\binom{2}{1}\binom{3}{1}}{28} = \frac{6}{28}$$

Similarly, $f(1,1) = P(X=1 \text{ and } Y=1)$

$= P(\text{1 is black, 1 is red and none is green})$

$$= \frac{\binom{3}{1}\binom{2}{1}\binom{3}{0}}{28} = \frac{6}{28}$$

Similar calculations give the probabilities of other values and the joint p.f. of X and Y is g

(x, y)	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$	$(0, 2)$	$(2, 0)$
$f(x, y)$	$\frac{3}{28}$	$\frac{6}{28}$	$\frac{9}{28}$	$\frac{6}{28}$	$\frac{1}{28}$	$\frac{3}{28}$

These probabilities can also be represented in another tabular form as follows:

Joint Probability Distribution

X \ Y	0	1	2	$P(X=x_i)$ $g(x)$
0	$\frac{3}{28}$	$\frac{6}{28}$	$\frac{1}{28}$	10/28
1	$\frac{9}{28}$	$\frac{6}{28}$	0	15/28
2	$\frac{3}{28}$	0	0	3/28
$P(Y=y_j)$ $h(y)$	15/28	12/28	1/28	1

this joint p.d. of the two r.v.'s (X, Y) can be represented by the formula.

$$f(x,y)=\frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{28},\qquad \begin{array}{l} x=0,1,2 \\ y=0,1,2 \\ 0\le x+y\le 2 \end{array}$$

To compute $P(X + Y \leq 1)$, we see that $x + y \leq 1$ for the cells (0, 0), (0, 1) and (1, 0). Therefore

$$P(X+Y \leq 1) = f(0,0) + f(0,1) + f(1,0)$$

$$= \frac{3}{28} + \frac{6}{28} + \frac{9}{28} = \frac{18}{28} = \frac{9}{14}$$

The marginal p.d.'s are

x	0	1	2
$g(x)$	10/28	15/28	3/28

y	0	1	2
h(y)	15/28	12/28	1/28

By definition the conditional p.d. $f(x \mid 1)$ is

$$f(x \mid 1) = P(X{=}x \mid Y{=}1)$$

$$= \frac{P(X = x \text{ and } Y = 1)}{P(Y = 1)} = \frac{f(x,1)}{h(1)}$$

$$h(1) = \sum_{x=0}^{2} f(x,1) = \frac{6}{28} + \frac{6}{28} + 0 = \frac{12}{28} = \frac{3}{7}$$

$$f(x|1) = \frac{f(x,1)}{h(1)} = \frac{7}{3} f(x,1), \; x = 0, 1, 2$$

$$f(0|1) = \frac{7}{3} f(0,1) = \left(\frac{7}{3}\right)\left(\frac{6}{28}\right) = \frac{1}{2}$$

$$f(1 \mid 1) = \frac{7}{3} f(1,1) = \left(\frac{7}{3}\right)\left(\frac{6}{28}\right) = \frac{1}{2}$$

$$f(2 \mid 1) = \frac{7}{3} f(2,1) = \left(\frac{7}{3}\right)(0) = 0$$

Hence the conditional p.d. of X given the $Y = 1$, is

x	0	1	2
$f(x \mid 1)$	1/2	1/2	0

v) Finally, $P(X=0 \mid Y=1) = f(0 \mid 1) = \dfrac{1}{2}$

vi) We find that $f(0, 1) = \dfrac{6}{28}$,

$$g(0) = \sum_{y=0}^{2} f(0, y) = \frac{3}{28} + \frac{6}{28} + \frac{1}{28} = \frac{10}{28}$$

$$h(1) = \sum_{x=0}^{2} f(x, 1) = \frac{6}{28} + \frac{6}{28} + 0 = \frac{12}{28}$$

Now $\dfrac{6}{28} \neq \dfrac{10}{28} \times \dfrac{12}{28}$,

i.e. $f(0, 1) \neq g(0)\, h(1)$,

and therefore X and Y are not statistically independent.

Example 7.7 The joint p.d. of two discrete r.v.'s X and Y is given by

$$f(x,y) = \frac{xy^2}{30} \qquad \text{for } x = 1, 2, 3 \text{ and } y = 1, 2.$$

Are X and Y independent?

The *marginal p.d.* for X is

$$\begin{aligned} g(x) &= \sum_y f(x,y) \\ &= \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x(1)^2}{30} + \frac{x(2)^2}{30} = \frac{x}{6}, \text{ for } x = 1, 2, 3; \end{aligned}$$

and the *marginal p.d.* for Y is

$$\begin{aligned} h(y) &= \sum_x f(x,y) \\ &= \sum_{x=1}^{3} \frac{xy^2}{30} = \frac{1y^2}{30} + \frac{2y^2}{30} + \frac{3y^2}{30} = \frac{y^2}{5}, \text{ for } y = 1, 2 \end{aligned}$$

$$\frac{xy^2}{30} = \frac{x}{6} \times \frac{y^2}{5}, \text{ for } x = 1, 2, 3 \text{ and } y = 1, 2,$$

i.e. $f(x, y) = g(x) \cdot h(y)$

X and Y are independent.

Continuous Random vector. The joint distribution of (X, Y) can be described via a nonnegative joint density function $f(x, y)$ such that for any subset $A \subset \mathbb{R}^2$,

$$P((X,Y) \in A) = \iint_A f(x,y)dxdy.$$

We should have

$$\iint_{\mathbb{R}^2} f(x,y)dxdy = 1.$$

Continuous random vector:

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, dudv$$

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

The *marginal* p.d.f. of the continuous r.v. X is

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

and of the r.v. Y is

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

That is the marginal p.d.f. of any of the variables is obtained by integrating out the other variable from the joint p.d.f. between the limits $-\infty$ and $+\infty$.

The *conditional* p.d.f. of the continuous r.v. X given that Y takes the value y, is defined to be

$$f(x|y) = \frac{f(x, y)}{h(y)},$$

where $f(x, y)$ and $h(y)$ are respectively the joint p.d.f. of X and Y, and the marginal p.d.f. of Y and $h(y)>0$.

Similarly, the *conditional* p.d.f. of the continuous r.v. Y given that $X = x$, is

$$f(y|x) = \frac{f(x, y)}{g(x)}, \text{ provided that } g(x) > 0.$$

Example 7.8 Given the following joint p.d.f.

$$f(x,y)=\frac{1}{8}(6-x-y),\ 0\le x\le 2;\ 2\le y\le 4,$$

$$=0, \text{ elsewhere.}$$

a) Verify that $f(x, y)$ is a joint density function.

b) Calculate (i) $P\left(X\le\frac{3}{2}, Y\le\frac{5}{2}\right)$, (ii) $P(X+Y<3)$.

c) Find the marginal p.d.f. $g(x)$ and $h(y)$.

d) Find the conditional p.d.f. $f(x\mid y)$ and $f(y\mid x)$.

a) The joint density $f(x, y)$ will be a p.d.f. if

i) $f(x, y)\ge 0$ and

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dx\,dy=1.$

- Now $f(x, y)$ is clearly ≥ 0 for all x and y in the given region, and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \frac{1}{8}\int_{0}^{2}\int_{2}^{4}(6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_{0}^{2}\left[6y - xy - \frac{y^2}{2}\right]_{2}^{4} dx$$

$$= \frac{1}{8}\int_{0}^{2}(6-2x)\,dx = \frac{1}{8}\left[6x - x^2\right]_{0}^{2}$$

$$= \frac{1}{8}[12-4] = 1.$$

Thus $f(x, y)$ has the properties of a joint p.d.f.

$$P(X \le \frac{3}{2}, Y \le \frac{5}{2}) = \int_{x=0}^{\frac{3}{2}} \int_{y=2}^{\frac{5}{2}} \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^{\frac{3}{2}} \left[6y - xy - \frac{y^2}{2}\right]_2^{\frac{5}{2}} dx$$

$$= \frac{1}{8}\int_0^{\frac{3}{2}} \left(\frac{15}{8} - \frac{x}{2}\right) dx = \frac{1}{64}\left[15x - 2x^2\right]_0^{\frac{3}{2}} = \frac{9}{32}$$

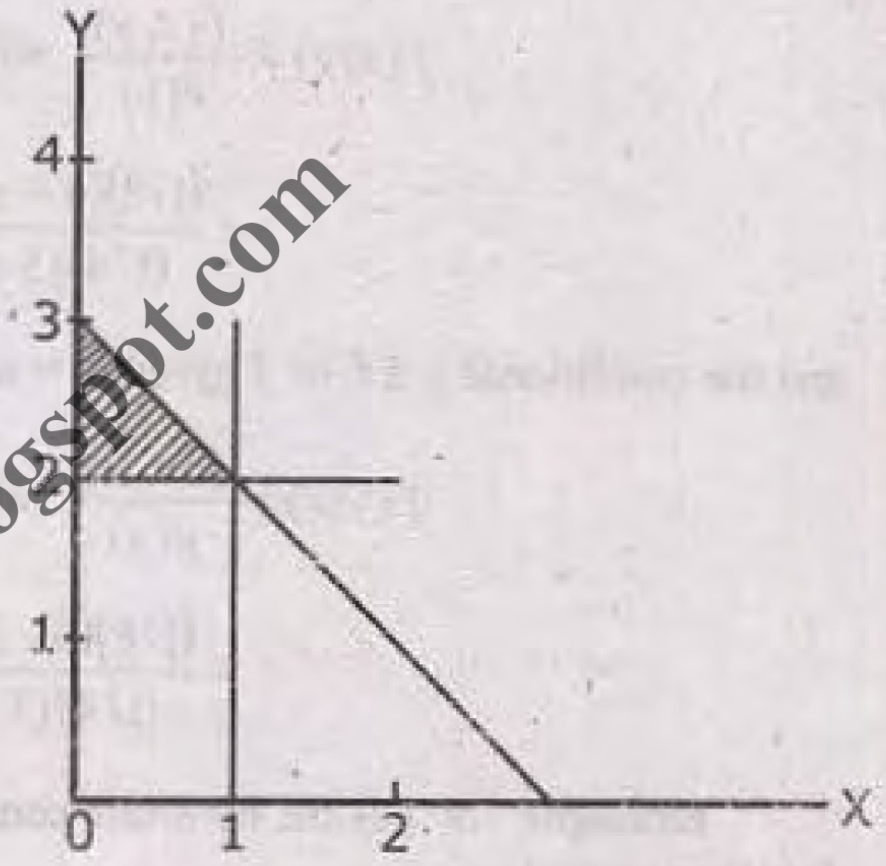
(ii) $P(X + Y < 3) = \frac{1}{8}\int_0^1 \int_2^{3-x} (6 - x - y)\,dy\,dx \; (\because x + y \le 3, \therefore y = 3 - x)$

$$= \frac{1}{8}\int_0^1 \left[6x - xy - \frac{y^2}{2}\right]_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^1 \left(\frac{x^2}{2} - 4x + \frac{7}{2}\right) dx$$

$$= \frac{1}{8}\left[\frac{x^3}{6} - 2x^2 - \frac{7x}{2}\right]_0^1$$

$$= \frac{1}{8} \times \frac{10}{6} = \frac{5}{24}$$



Event "$X + Y \le 3$" is shaded

The marginal p.d.f. of X is

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \qquad -\infty < x < \infty$$

$$= \frac{1}{8}\int_{2}^{4}(6-x-y)\,dy, \qquad 0 \le x \le 2$$

$$= \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4 \qquad 0 \le x \le 2.$$

$$= \frac{1}{4}(3-x) \qquad 0 \le x \le 2.$$

$$= 0 \qquad x < 0 \; or \; x \ge 2.$$

Similarly, the marginal p.d.f. of Y is

$$h(y) = \frac{1}{8}\int_0^2 (6-x-y)\,dx, \qquad 2 \le y \le 4$$

$$= \frac{1}{4}(5-y) \qquad 2 \le y \le 4$$

$$= 0, \text{ elsewhere.}$$

d) The conditional p.d.f. of X given Y = y, is

$$f(x|y) = \frac{f(x,y)}{h(y)}, \text{ where } h(y) > 0.$$

$$= \frac{(1/8)(6-x-y)}{(1/4)(5-y)} = \frac{6-x-y}{2(5-y)}$$

and the conditional p.d.f. of Y given X = x, is

$$f(y|x) = \frac{f(x,y)}{g(x)}, \text{ where } g(x) > 0.$$

$$= \frac{(1/8)(6-x-y)}{(1/4)(3-x)} = \frac{6-x-y}{2(3-x)}.$$

Mathematical Expectation:

Discrete r.v's:

Let be a discrete r.v, then its expected value is denoted by and is defined as:

- Example: Tossing 3 coins: X H

	0	1	2	3
	1/8	3/8	3/8	1/8

a) If its rain, an umbrella salesman can earn $ 30 per day. If it is fair, he can lose $ 6 per day. What is his expectation if the probability of rain is 0.3?

expected rain= 30 and probability 0.3

rain not expectd=-6 and probability=0.7

so 30*0.3=**9**

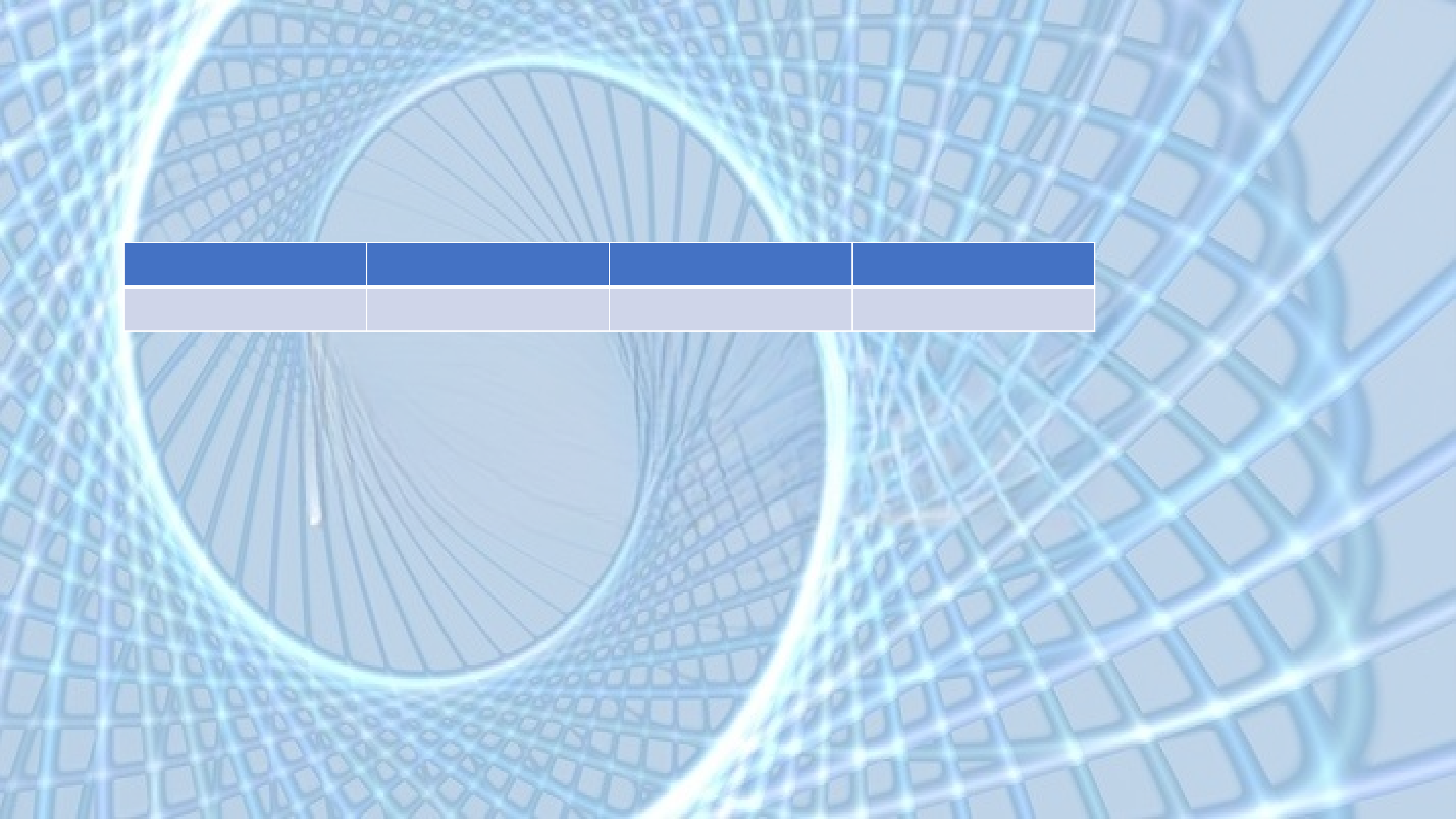
and -6*0.7=**-4.2**

=E(X)=9-4.2=4.8$ per day

Example:

- b) A man draws 2 balls from a bag containing 3 white and 5 black balls. If he receives Rs. 70 for white balls he draw and Rs. 7 for black ball he draws. What is the expected value.

- Bag contains 3 W and 5 B   Total number of balls = 8
- A man picks 2 Balls
- Possibilities are : 2 W,   1W and 1B,   2B balls may be selected.
- Probabilities:



Continuous r.vs:

Let be a continuous r.v with pdf then its expected value will be defined as:

- Example: Let be a continuous r.v with pdf

Expectation of function of a rv:

- Discrete rv: Let be rv having pd , and let be a function of rv , Then
- Continuous rv: : Let be a cont rv having pdf , and let be a function of rv , Then

Median and Mode:

- Median: If is a continuous rv: Then a number will be median if

- Quartiles:

- Mode: For a continuous random variable having pdf A stationary point of will be a mode of if

Example 7.13 Let X have the following probability distribution:

x_i	1	2	3	4	5
$f(x_i)$	0.2	0.3	0.2	0.2	0.1

Find the probability functions of $3X - 1$, X^2 and $X^2 + 2$; and find $E(3X - 1)$, $E(X^2)$ and $E(X^2 + 2)$.

The probability distribution of the r.v. $H(X) = 3X - 1$, is

Values of X, x_i	1	2	3	4	5
Probabilities, $f(x_i)$	0.2	0.3	0.2	0.2	0.1
Values of $3X-1$, $(3x_i-1)$	2	5	8	11	14

$$E(3X-1) = \Sigma H(x_i)f(x_i) = \Sigma(3x_i - 1)f(x_i)$$

$$= 2 \times 0.2 + 5 \times 0.3 + 8 \times 0.2 + 11 \times 0.2 + 14 \times 0.1$$

$$= 0.4 + 1.5 + 1.6 + 2.2 + 1.4 = 7.1$$

The p.d. of $H(X) = X^2$ is

x_i	1	2	3	4	5
$f(x_i)$	0.2	0.3	0.2	0.2	0.1
x_i^2	1	4	9	16	25

and $\quad E(X^2) = \sum x_i^2 f(x_i)$

$$= 1 \times 0.2 + 4 \times 0.3 + 9 \times 0.2 + 16 \times 0.2 + 25 \times 0.1$$

$$= 0.2 + 1.2 + 1.8 + 3.2 + 2.5 = 8.9$$

Similarly, $\quad E(X^2 + 2) = \sum (x_i^2 + 2) f(x_i)$

$$= 3 \times 0.2 + 6 \times 0.3 + 11 \times 0.2 + 18 \times 0.2 + 27 \times 0.1$$

$$= 0.6 + 1.8 + 2.2 + 3.6 + 2.7 = 10.9$$

Example 7.14 Let X be a r.v. with p.d.f.

$$f(x) = 2(x-1), \qquad 1 < x < 2$$
$$= 0, \qquad \text{elsewhere.}$$

Find the expected values of $H(X) = 2X - 1$ and $H(X) = X^2$.

$$\text{Now} \quad E(2X-1) = \int_{-\infty}^{\infty}(2x-1)\,f(x)\,dx = 2\int_{1}^{2}(2x-1)(x-1)\,dx$$

$$= 2\int_{1}^{2}(2x^2-3x+1)\,dx = 2\left[\frac{2x^3}{3}-\frac{3x^2}{2}+x\right]_1^2$$

$$= 2\left[\left(\frac{16}{3}-6+2\right)-\left(\frac{2}{3}-\frac{3}{2}+1\right)\right]$$

$$= 2\left[\frac{4}{3}-\frac{1}{6}\right] = \frac{7}{3}$$

and $\quad E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= 2\int_1^2 x^2(x-1)\,dx = 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$$

$$= 2\left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right]$$

$$= 2\left[\frac{4}{3} + \frac{1}{12}\right] = \frac{17}{6}$$

Example 7.15 If the continuous r.v. X has p.d.f.

$$f(x) = \frac{3}{4}(3-x)(x-5), \quad 3 \le x \le 5$$

$$= 0, \qquad \text{elsewhere.}$$

ate the arithmetic mean, variance and standard deviation of X.

Now $\quad E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \frac{3}{4}\int_{3}^{5} x.(3-x)(x-5)\, dx$$

$$= \frac{3}{4}\int_{3}^{5} (-x^3 + 8x^2 - 15x)\, dx = \frac{3}{4}\left[-\frac{x^4}{4} + \frac{8x^3}{3} - \frac{15x^2}{2}\right]_3^5$$

$$= \frac{3}{4}\left[\left(-\frac{625}{4} + \frac{1000}{3} - \frac{375}{2}\right) - \left(-\frac{81}{4} + \frac{216}{3} - \frac{135}{2}\right)\right]$$

Example 7.15 If the continuous r.v. X has p.d.f.

$$f(x) = \frac{3}{4}(3-x)(x-5), \quad 3 \le x \le 5$$

$$= 0, \qquad \text{elsewhere.}$$

late the arithmetic mean, variance and standard deviation of X.

$$\text{Now} \quad E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{3}{4}\int_{3}^{5} x.(3-x)(x-5)\, dx$$

$$= \frac{3}{4}\int_{3}^{5}(-x^3 + 8x^2 - 15x)\, dx = \frac{3}{4}\left[-\frac{x^4}{4} + \frac{8x^3}{3} - \frac{15x^2}{2}\right]_3^5$$

$$= \frac{3}{4}\left[\left(-\frac{625}{4} + \frac{1000}{3} - \frac{375}{2}\right) - \left(-\frac{81}{4} + \frac{216}{3} - \frac{135}{2}\right)\right]$$

$$\therefore \quad Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{81}{5} - (4)^2 = \frac{1}{5} = 0.2, \text{ and}$$

$$S.D.(X) = \sqrt{0.2} = 0.447$$

mean = 4, variance = 0.2 an standard deviation = 0.447.

Example 7.16 The continuous *r.v.* X has p.d.f. $f(x)$, where $f(x) = \frac{3}{4}(1+x^2)$ for $0 \le x \le 1$

$E(X) = \mu$ and $Var(X) = \sigma^2$, find $P(|X-\mu| < \sigma)$.

Now $E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$

$$= \frac{3}{4}\int_0^1 x(1+x^2)\,dx = \frac{3}{4}\int_0^1 (x+x^3)\,dx$$

$$= \frac{3}{4}\left[\frac{x^2}{2} + \frac{x^4}{4}\right]_0^1 = \frac{3}{4}\left[\frac{1}{2} + \frac{1}{4}\right] = \frac{9}{16} = 0.5625$$

And $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \frac{3}{4}\int_0^1 x^2(1+x^2)\,dx = \frac{3}{4}\int_0^1 (x^2+x^4)\,dx$$

$$= \frac{3}{4}\left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4}\left[\frac{1}{3} + \frac{1}{5}\right]$$

$$= \frac{3}{4}\left(\frac{8}{15}\right) = \frac{2}{5} = 0.4$$

$$\therefore \ Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{5} - \left(\frac{9}{16}\right)^2$$

$$= \frac{107}{1280} = 0.0836, \text{ so that}$$

S.D. (X) or $\sigma = \sqrt{0.0836} = 0.289$

Now $P(|X - \mu| < \sigma) = P(-\sigma < X - \mu < \sigma)$

$$= P(\mu - \sigma < X < \mu + \sigma)$$

$$= P(0.5625 - 0.289 < X < 0.5625 + 0.289)$$

$$= P(0.2735 < X < 0.8515), \text{ and}$$

$$P(0.2735 < X < 0.8515) = \frac{3}{4}\int_{0.2735}^{0.8515}(1+x^2)\,dx$$

$$= \frac{3}{4}\left[x + \frac{x^3}{3}\right]_{0.2735}^{0.8515}$$

$$= \frac{3}{4}\left[0.8515 + \frac{(0.8515)^3}{3} - \left(0.2735 + \frac{(0.2735)^3}{3}\right)\right]$$

$$= \frac{3}{4}[1.05729 - 0.28032] = 0.8527$$

... $P(|X-\mu| < \sigma) = 0.8527.$

Example 7.17 A continuous *r.v.* X has the *p.d.f.*

$$f(x) = \frac{3}{4}x(2-x), \quad 0 \le x \le 2.$$

$$= 0, \qquad \text{elsewhere}$$

...the first four moments about the mean and the coefficient of skewness.

We first calculate the moments about origin as:

$$\mu_1' = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x(2x - x^2)\,dx = \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4}\left[\frac{16}{12}\right] = 1;$$

$$\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^2(2x - x^2)\,dx = \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[8 - \frac{32}{5}\right] = \frac{3}{4}\left[\frac{8}{5}\right] = \frac{6}{5};$$

$$\mu_3' = E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^3(2x - x^2)\,dx = \frac{3}{4}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$=\frac{3}{4}\left[\frac{64}{5}-\frac{64}{6}\right]=\frac{3}{4}\left[\frac{64}{30}\right]=\frac{8}{5};$$

$$\mu_4' = E(X^4) = \int_{-\infty}^{\infty} x^4 f(x)\,dx$$

$$=\frac{3}{4}\int_0^2 x^4(2x-x^2)\,dx = \frac{3}{4}\left[\frac{2x^6}{6}-\frac{x^7}{7}\right]_0^2$$

$$=\frac{3}{4}\left[\frac{64}{3}-\frac{128}{7}\right]=\frac{3}{4}\left[\frac{64}{21}\right]=\frac{16}{7}.$$

Then we find the moments about the mean as;

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{6}{5} - (1)^2 = \frac{1}{5}$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

$$= \frac{8}{5} - 3(1)\left(\frac{6}{5}\right) + 2(1)^3 = \frac{8}{5} - \frac{18}{5} + 2 = 0;$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4$$

$$= \frac{16}{7} - 4(1)\left(\frac{8}{5}\right) + 6(1)^2\left(\frac{6}{5}\right) - 3(1)^4$$

$$= \frac{16}{7} - \frac{32}{5} + \frac{36}{5} - 3 = \frac{3}{35}$$

The coefficient of skewness is

$$\gamma_1 = \frac{E(X-\mu)^3}{\sigma^3} = 0 \qquad [\because E(X-\mu)^3 = \mu_3]$$

Geometric and Hormonic Means:

- Geometric Mean:

- Hormonic Mean:

Example 7.24 Let X be a r.v. with p.d.f.

$$f(x) = k\,(x - x^2), \qquad 0 \le x \le 1,$$
$$= 0, \qquad \text{elsewhere,}$$

where k is a constant. Then find the mean, median, mode, harmonic mean and standard deviati

First of all, we find the value of k which should be such as to make

$$\int_0^1 k(x - x^2)\,dx = 1$$

That is $k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$ or $k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$ or $k = 6$.

Now, the mean, μ or $E(X)$ is given by

$$\mu = E(X) = \int_0^1 x.6(x-x^2)\,dx$$

$$= 6\int_0^1 (x^2 - x^3)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

The median, a, is given by $\int_{-\infty}^{a} f(x)\,dx = \frac{1}{2}$. Thus

$$6\int_0^a (x - x^2)\,dx = \frac{1}{2} \text{ or } 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{1}{2}$$

or $\quad 4a^3 - 6a^2 + 1 = 0$

factorize the equation $4a^3 - 6a^2 + 1 = 0$ as

$$(2a-1)(2a^2-2a-1) = 0$$

either $2a - 1 = 0$ which gives $a = \frac{1}{2}$, or

$a^2 - 2a - 1 = 0$, which gives $a = \frac{1 \pm \sqrt{3}}{2}$, i.e. $a = -0.366$ or 1.366.

The values -0.366 and 1.366 are unacceptable since a must lie in the interval $(0, 1)$. The median is given by $a = \frac{1}{2}$.

Mode is that value of x for which

(i) $f'(x) = 0$, and (ii) $f''(x) < 0$

$f(x) = 6(x - x^2)$, and $f'(x) = 6(1 - 2x)$

$f'(x) = 0$ when $1 - 2x = 0$ or $x = \frac{1}{2}$

To check that this is maximum, we find

$$f''(x) = 6(-2) = -12 < 0$$

i.e. $f''(x)$ is negative for all values of x, there is a maximum at $x = \frac{1}{2}$

Hence the mode $= \frac{1}{2}$.

The harmonic mean, H, is given by

$$\frac{1}{H} = \int_0^1 \frac{1}{x}.6(x - x^2)\,dx$$

$$= 6\int_0^1 (1-x)\,dx = 6\left[x - \frac{x^2}{2}\right]_0^1 = 3$$

$$\therefore \quad H = \frac{1}{3}$$

Again, $\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \int_0^1 x^2 .6(x - x^2)\,dx = 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{3}{10}$$

$$\therefore \quad \mu_2 = E(X^2) - [E(X)]^2 = \mu_2' - \mu_1'^2$$

$$= \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20} = 0.05$$

Hence $\sigma = \sqrt{\mu_2} = \sqrt{0.05} = 0.2234$

Moment Generating Function

if is continuous rv.

If is discrete rv.

Quiz # 2 10 min (Data Science)P

- 1- If a committee of size 3 selected from 7 men and 5 women. What is the probability that it will contain exactly 2 men?

- In a bolt factory, three machines M_1, M_2, and M_3 manufacture 2000, 2500, and 4000 bolts every day. Of their output 3%, 4%, and 2.5% are defective bolts. One of the bolts is drawn very randomly from a day's production and is found to be defective. What is the probability that it was produced by machine M_2?

Quiz #2 T 10 min

- 1- 5 books are selected from 8 Maths and 7 Physics books, what is the probability that all of them are Maths books?

- 2- 1% of a population have a certain disease and the remaining 99% are free from this disease. A test is used to detect this disease. This test is positive in 95% of the people with the disease and is also (falsely) positive in 2% of the people free from the disease. If a person, selected at random from this population, has tested positive, what is the probability that she/he has the disease?

Quiz # 2 U 10 min

1. 4-digit numbers are formed from 1,2,3,4,5,6 without repetition and then a number is selected at random, what is probability that it will be divisible by 5?
2. Two production lines produce the same part. Line 1 produces 1,000 parts per week of which 100 are defective. Line 2 produces 2,000 parts per week of which 150 are defective. If you choose a part randomly from the stock what is the probability it is defective? If it is defective what is the probability it was produced by line 1?

Quiz # 2 E 10 min

1. 4-digit numbers are formed from 1,2,3,4,5,6 without repetition and then a number is selected at random, what is probability that it will not be divisible by 5?
2. 1% of a population have a certain disease and the remaining 99% are free from this disease. A test is used to detect this disease. This test is positive in 95% of the people with the disease and is also (falsely) positive in 2% of the people free from the disease. If a person, selected at random from this population, has tested positive, what is the probability that she/he has the disease?